Appendix C: NURBS

- *NURBS* ("*Non-Uniform Rational B-Splines*") are a generalization of Beziers.
  - NU: *Non-Uniform*. The knots in the knot vector are not required to be uniformly spaced.
  - R: *Rational*. The spline may be defined by rational polynomials (homogeneous coordinates.)
  - BS: *B-Spline*. A generalized Bezier spline with controllable degree.

B-Splines

We'll build our definition of a B-spline from:

- d, the *degree* of the curve
- $k = d+1$, called the *parameter* of the curve
- $\{P_1 \ldots P_n\}$, a list of n *control points*
- $[t_1, \ldots, t_{k+n}]$, a *knot vector* of (k+n) parameter values ("knots")
- $d = k-1$ is the degree of the curve, so k is the number of control points which influence a single interval.
  - Ex: a cubic (d=3) has four control points (k=4).
- There are $k+n$ knots t_i, and $t_i \leq t_{i+1}$ for all t_i.
- Each B-spline is $C^{(k-2)}$ continuous: *continuity* is degree minus one, so a k=3 curve has d=2 and is C1.

B-Splines

- The equation for a B-spline curve is

$$P(t) = \sum_{i=1}^{n} N_{i,k}(t) P_i, \; t_{min} \le t < t_{max}$$

- $N_{i,k}(t)$ is the *basis function* of control point P_i for parameter k. $N_{i,k}(t)$ is defined recursively:

$$N_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

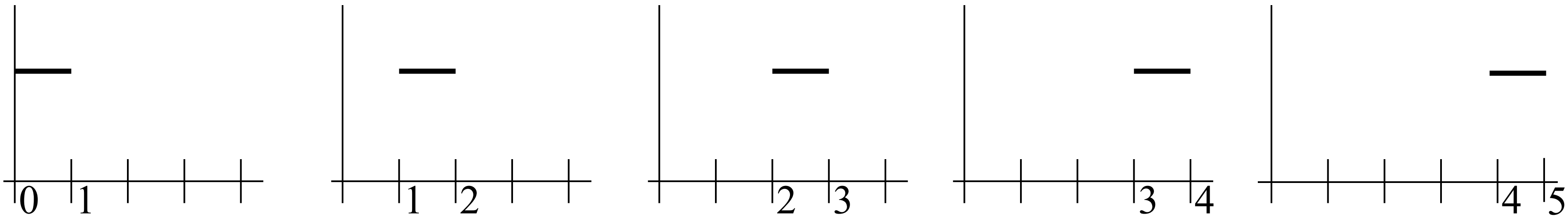
B-Splines

$t_1 \qquad t_2 \qquad t_3 \qquad t_4 \qquad t_5 \quad \ldots$

$$
\begin{tikzcd}[column sep=tiny]
N_{1,1}(t) \arrow[dr] & & N_{2,1}(t) \arrow[dl] \arrow[dr] & & N_{3,1}(t) \arrow[dl] \arrow[dr] & & N_{4,1}(t) \arrow[dl] \arrow[dr] & & \ldots \\
& N_{1,2}(t) \arrow[dr] & & N_{2,2}(t) \arrow[dl] \arrow[dr] & & N_{3,2}(t) \arrow[dl] \arrow[dr] & & \ldots & \\
& & N_{1,3}(t) \arrow[dr] & & N_{2,3}(t) \arrow[dl] \arrow[dr] & & \ldots & & \\
& & & N_{1,4}(t) & & \ldots & & &
\end{tikzcd}
$$

$$N_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

B-Splines



$N_{1,1}(t)$ $N_{2,1}(t)$ $N_{3,1}(t)$ $N_{4,1}(t)$ $N_{5,1}(t)$

$t_1 = 0.0$
$t_2 = 1.0$
$t_3 = 2.0$
$t_4 = 3.0$
$t_5 = 4.0$
$t_6 = 5.0$

$N_{1,1}(t)=1,\ 0 \le t < 1$

$N_{3,1}(t)=1,\ 2 \le t < 3$

$N_{5,1}(t)=1,\ 4 \le t < 5$

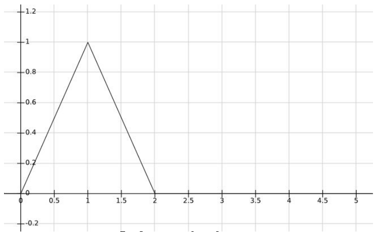
$N_{2,1}(t)=1,\ 1 \le t < 2$

$N_{4,1}(t)=1,\ 3 \le t < 4$

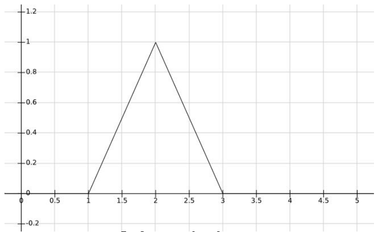
Knot vector = $\{0,1,2,3,4,5\}$, $k = 1 \rightarrow d = 0$ (degree = zero)

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

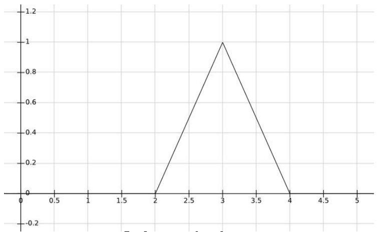
B-Splines



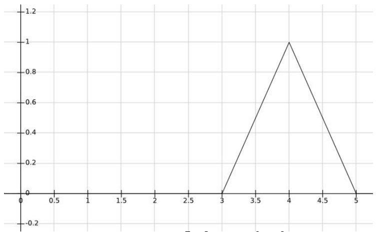
$N_{1,2}(t)$



$N_{2,2}(t)$



$N_{3,2}(t)$



$N_{4,2}(t)$

$$N_{1,2}(t) = \frac{t-0}{1-0} N_{1,1}(t) + \frac{2-t}{2-1} N_{2,1}(t) = \begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \end{cases}$$

$$N_{2,2}(t) = \frac{t-1}{2-1} N_{2,1}(t) + \frac{3-t}{3-2} N_{3,1}(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 3-t & 2 \le t < 3 \end{cases}$$

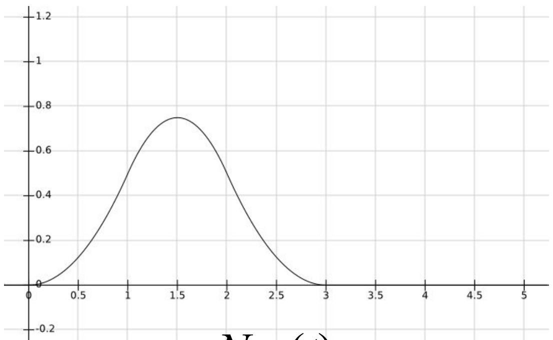
$$N_{3,2}(t) = \frac{t-2}{3-2} N_{3,1}(t) + \frac{4-t}{4-3} N_{4,1}(t) = \begin{cases} t-2 & 2 \le t < 3 \\ 4-t & 3 \le t < 4 \end{cases}$$

$$N_{4,2}(t) = \frac{t-3}{4-3} N_{4,1}(t) + \frac{5-t}{5-4} N_{5,1}(t) = \begin{cases} t-3 & 3 \le t < 4 \\ 5-t & 4 \le t < 5 \end{cases}$$

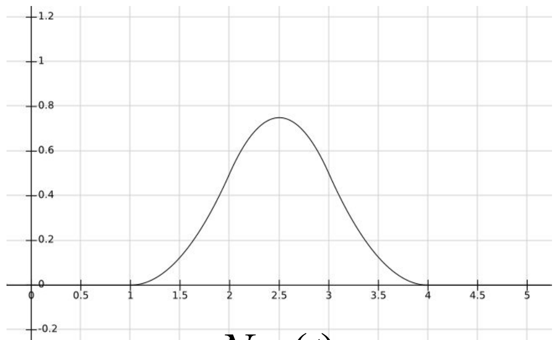
Knot vector = {0,1,2,3,4,5}, $k = 2 \rightarrow d = 1$ (degree = one)

B-Splines

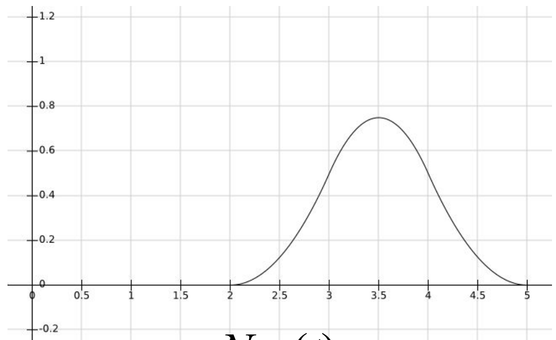
$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$



$N_{1,3}(t)$



$N_{2,3}(t)$



$N_{3,3}(t)$

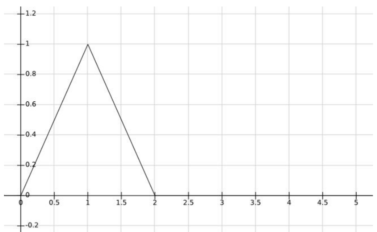
$$N_{1,3}(t) = \frac{t-0}{2-0} N_{1,2}(t) + \frac{3-t}{3-1} N_{2,2}(t) = \begin{cases} t^2/2 & 0 \le t < 1 \\ -t^2 + 3t - 3/2 & 1 \le t < 2 \\ (3-t)^2/2 & 2 \le t < 3 \end{cases}$$

$$N_{2,3}(t) = \frac{t-1}{3-1} N_{2,2}(t) + \frac{4-t}{4-2} N_{3,2}(t) = \begin{cases} (t-1)^2/2 & 1 \le t < 2 \\ -t^2 + 5t - 11/2 & 2 \le t < 3 \\ (4-t)^2/2 & 3 \le t < 4 \end{cases}$$

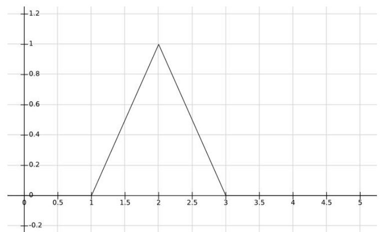
$$N_{3,3}(t) = \frac{t-2}{4-2} N_{3,2}(t) + \frac{5-t}{5-3} N_{4,2}(t) = \begin{cases} (t-2)^2/2 & 2 \le t < 3 \\ -t^2 + 7t - 23/2 & 3 \le t < 4 \\ (5-t)^2/2 & 4 \le t < 5 \end{cases}$$

Knot vector = {0,1,2,3,4,5}, $k = 3 \rightarrow d = 2$ (degree = two)

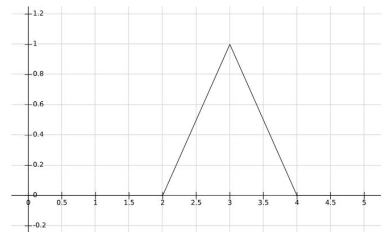
Basis functions really sum to one (k=2)



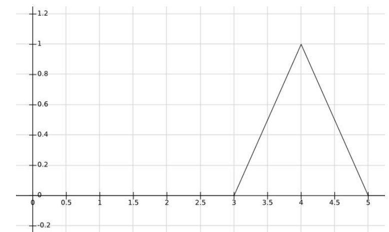
$N_{1,2}(t)$



$N_{2,2}(t)$

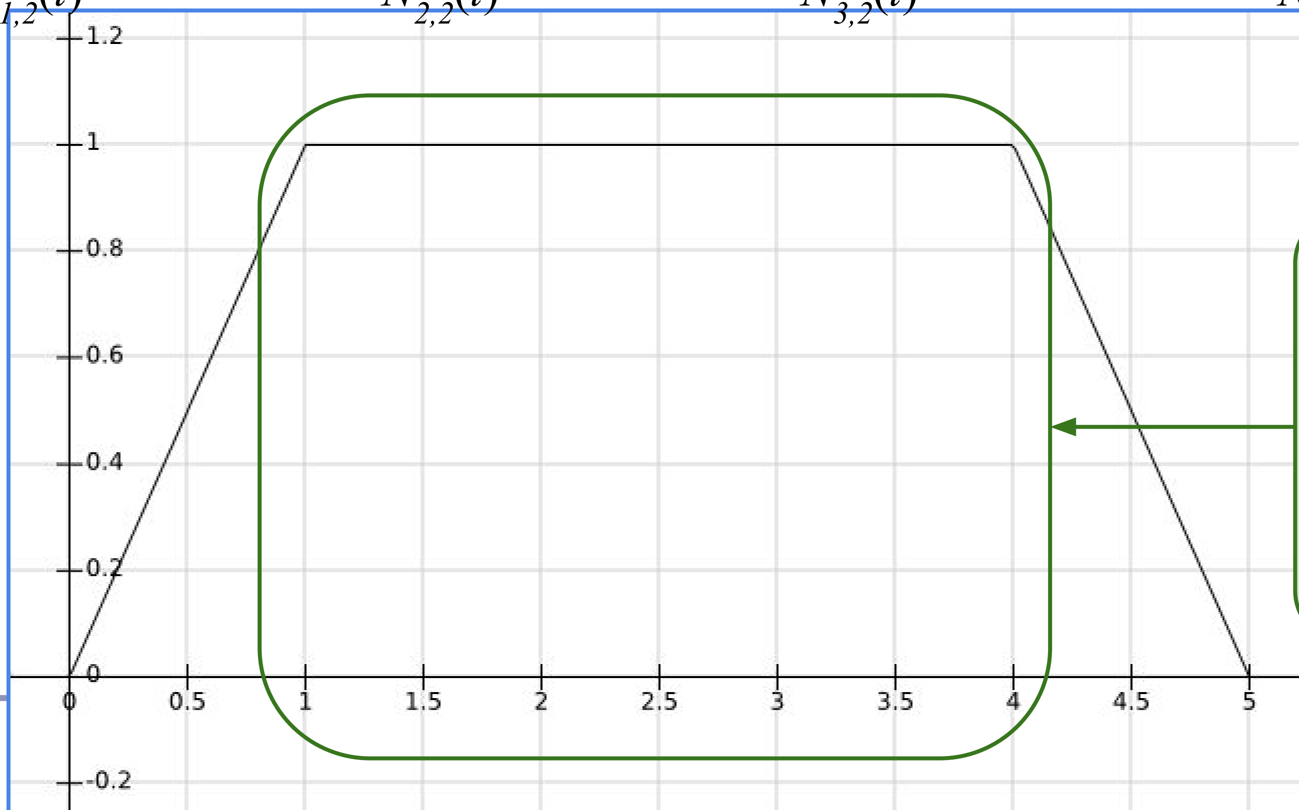


$N_{3,2}(t)$



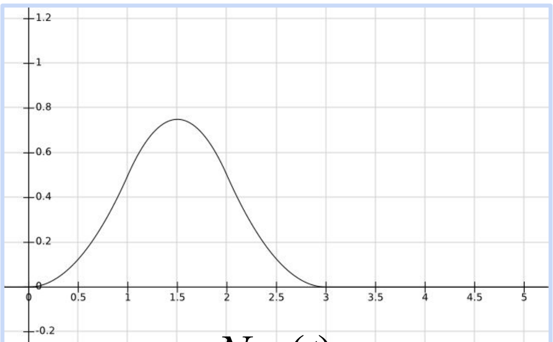
$N_{4,2}(t)$

=



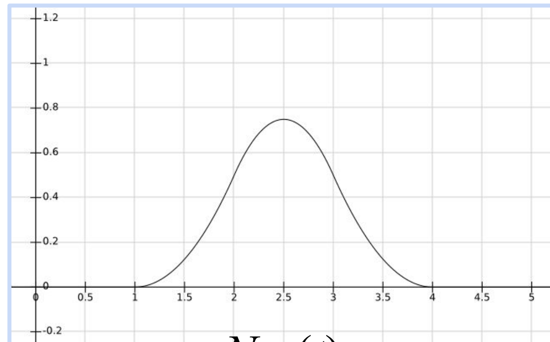
The sum of the four basis functions is fully defined (sums to one) between t_2 (t=1.0) and t_5 (t=4.0).

Basis functions really sum to one (k=3)



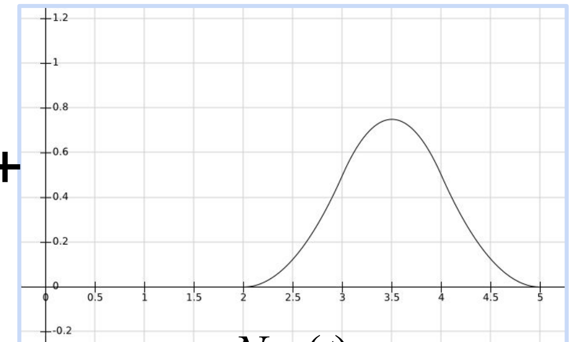
$N_{1,3}(t)$

+



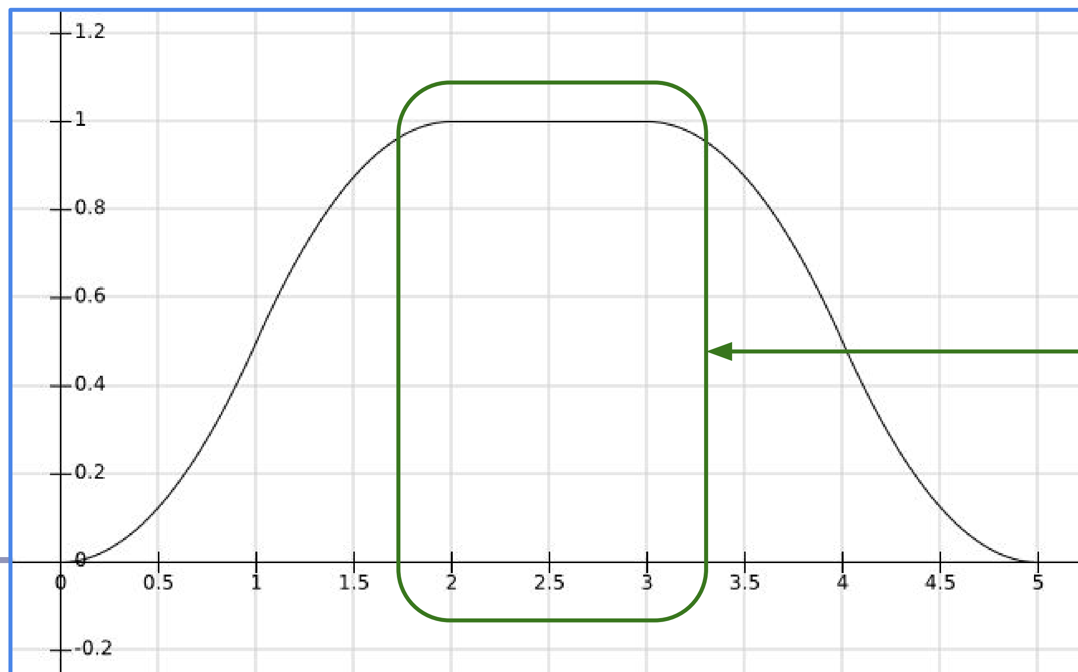
$N_{2,3}(t)$

+



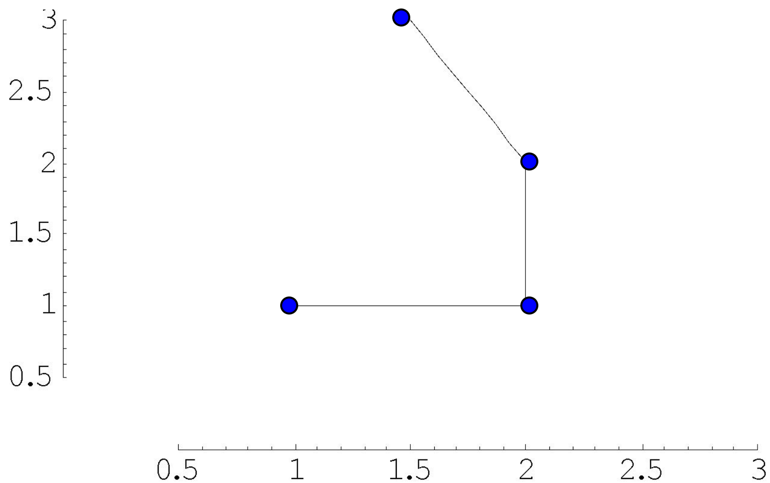
$N_{3,3}(t)$

=

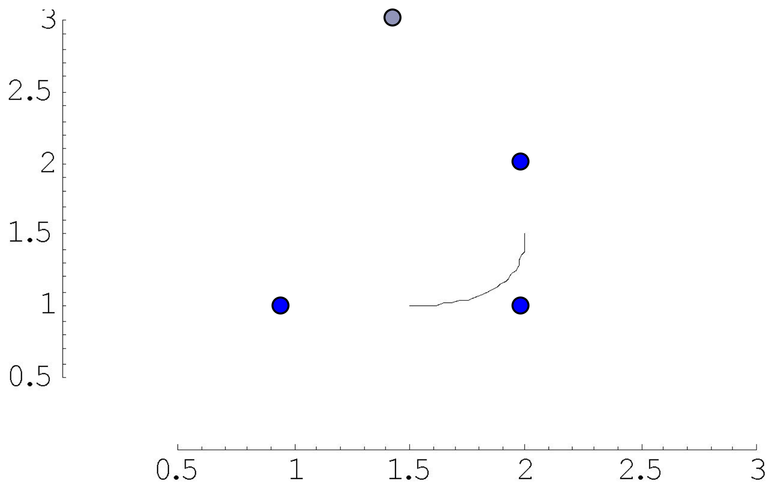


The sum of the three functions is fully defined (sums to one) between t_3 (t=2.0) and t_4 (t=3.0).

B-Splines



At k=2 the function is piecewise linear, depends on P_1,P_2,P_3,P_4, and is fully defined on $[t_2, t_5)$.



At k=3 the function is piecewise quadratic, depends on P_1,P_2,P_3, and is fully defined on $[t_3, t_4)$.

Each parameter-k basis function depends on $k+1$ knot values; $N_{i,k}$ depends on t_i through t_{i+k}, inclusive. So six knots → five discontinuous functions → four piecewise linear interpolations → three quadratics, interpolating three control points. n=3 control points, d=2 degree, k=3 parameter, n+k=6 knots.

Knot vector = {0,1,2,3,4,5}

Non-Uniform B-Splines

- The knot vector {0,1,2,3,4,5} is *uniform*:

  $$t_{i+1}-t_i = t_{i+2}-t_{i+1} \ \forall\, t_i.$$

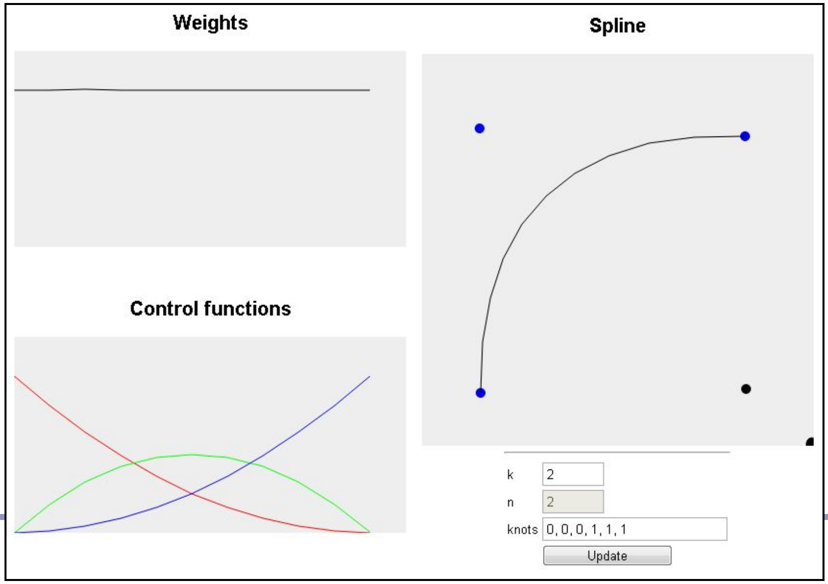
- Varying the size of an interval changes the parametric-space distribution of the weights assigned to the control functions.
- Repeating a knot value reduces the continuity of the curve in the affected span by one degree.
- Repeating a knot *k* times will lead to a control function being influenced <u>only</u> by that knot value; the spline will pass through the corresponding control point with C0 continuity.

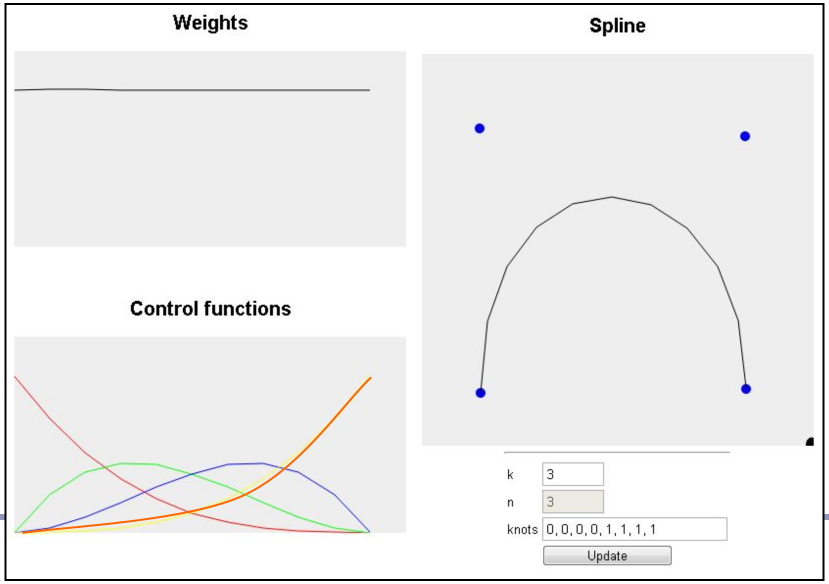
Open vs *Closed*

- A knot vector which repeats its first and last knot values k times is called *open*, otherwise *closed*.
  - Repeating the knots k times is the only way to force the curve to pass through the first or last control point.
  - Without this, the functions $N_{1,k}$ and $N_{n,k}$ which weight P_1 and P_n would still be 'ramping up' and not yet equal to one at the first and last t_i.

Open vs Closed

- Two examples you may recognize:
  - k=3, n=3 control points, knots={0,0,0,1,1,1}
  - k=4, n=4 control points, knots={0,0,0,0,1,1,1,1}

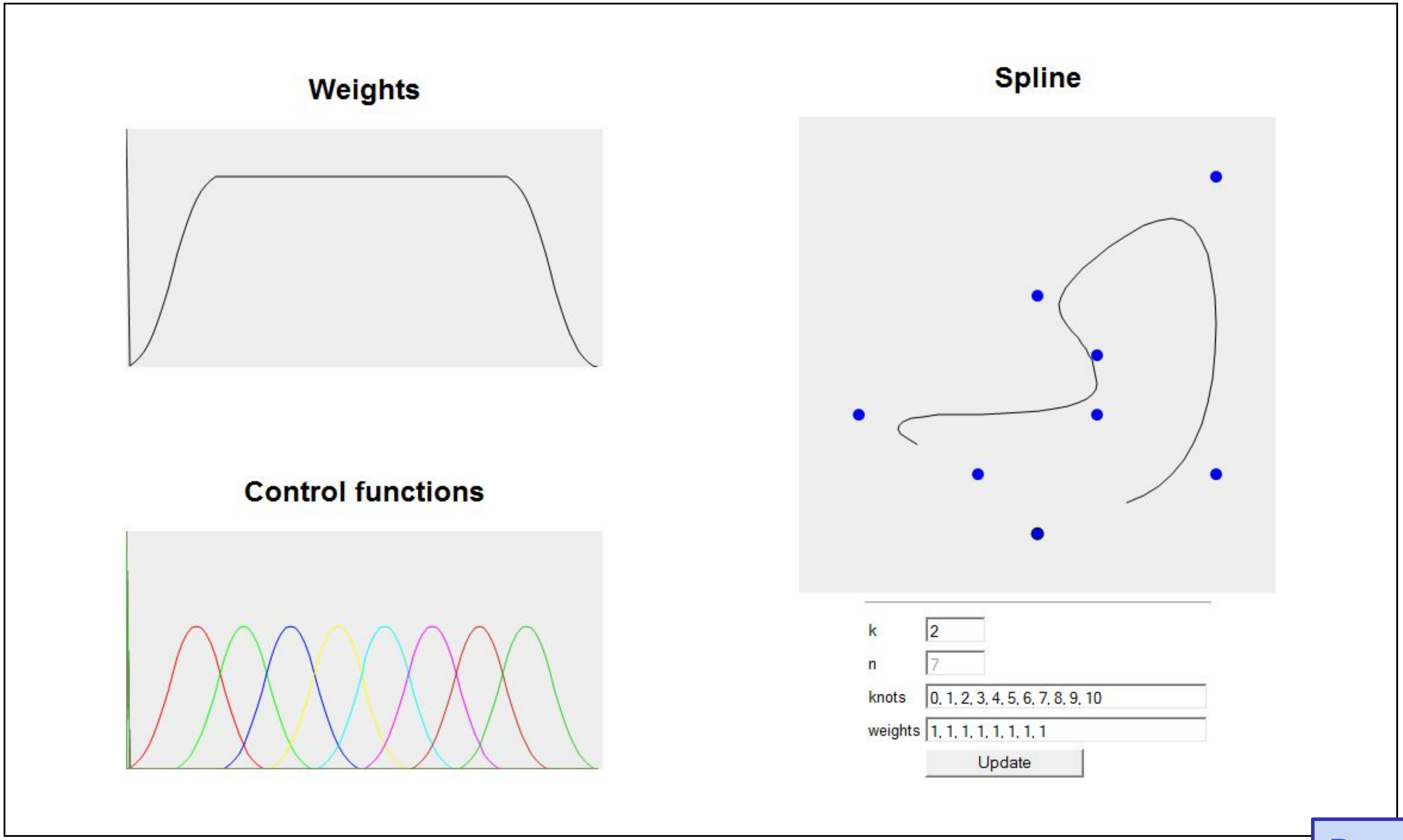




Non-Uniform *Rational* B-Splines

- Repeating knot values is a clumsy way to control the curve's proximity to the control point.
  - The solution: *homogeneous coordinates*.
  - Associate a 'weight' with each control point, ω_i, so that the expression becomes a weighted average
  - This allows us to slide the curve nearer or farther to individual control points without losing continuity or introducing new control points.

Non-Uniform Rational B-Splines in action



[Demo]

NURBS - References

- Les Piegl and Wayne Tiller, *The NURBS Book*, Springer (1997)
- Alan Watt, *3D Computer Graphics*, Addison Wesley (2000)
- G. Farin, J. Hoschek, M.-S. Kim, *Handbook of Computer Aided Geometric Design*, North-Holland (2002)